DECOMPOSING PAIRS OF MODULES[1]

BY
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Since every module can be represented as a homomorphic image $U = f(M)$ of a projective module M, it seems natural to enquire under what conditions a decomposition

$$U = U_1 \oplus \cdots \oplus U_n \tag{1}$$

arises from a decomposition of the entire representation:

$$M = M_1' \oplus \cdots \oplus M_n' \quad \text{with} \quad f(M_i') = U_i. \tag{2}$$

Further, we can ask for which decompositions $M = M_1 \oplus \cdots \oplus M_n$ can we choose $M_i' \cong M_i$ in (2)?

We will call an R-module U *semiprimary* if the ring of multiplications of R on $U/\text{rad } U$ is semisimple with minimum condition and U is finitely generated. (All modules considered in this paper will be unitary.) Our Main Theorem 1.2 states that if $f: M = M_1 \oplus \cdots \oplus M_n$ onto $U = U_1 + \cdots + U_n$ is a homomorphism of R-modules with M projective and each U_i semiprimary, and if, for each i there is *some* homomorphism of M_i onto U_i, then there is a decomposition (2) for which $M_i' \cong M_i$. Note that the sum $U_1 + \cdots$ need *not* be direct.

The main theorem is applied to sharpen a theorem of Steinitz, Chevalley, and others [7, Theorem 22.12] which states that if N is a submodule of a finitely generated, torsion-free (hence projective) module M over a Dedekind domain R such that M/N is a torsion module, then there exist decompositions $M = M_1 \oplus \cdots \oplus M_n$ ($M_i \cong$ an ideal of R) and $N = E_1M_1 \oplus \cdots \oplus E_nM_n$ ($E_i =$ an ideal). The present contribution to this theorem (Corollary 1.10) is that $M_1, \cdots, M_{n-1}$ can be chosen isomorphic to arbitrary nonzero ideals and the ideals E_i can be chosen subject only to the restriction $M/N \cong R/E_1 \oplus \cdots \oplus R/E_n$. This sharpened version of the theorem turns out to have a more abstract (and slightly more general) formulation (Corollary 1.9): if N_i is a submodule of a projective module M_i (still over a Dedekind domain) such that $M_1/N_1 \cong M_2/N_2 =$ finitely generated, then $M_1 \cong M_2$ if and only if $N_1 \cong N_2$; and when the conditions

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hold there is an isomorphism of M_1 onto M_2 which carries N_1 onto N_2. We show that both the classical and abstract forms of this theorem can be reformulated in the more general context of projective modules of rank 1 over a commutative ring, and that it is possible (in this more general context) to consider simultaneously a module and several of its submodules (Theorem 1.6 and Remarks 1.11).

A curious consequence of the main theorem (Corollary 1.12) is: given a homomorphism of a ring onto a semiprimary ring, if some finite orthogonal set of idempotents can be lifted, then so can every orthogonal set of idempotents isomorphic to them.

If the ring R has minimum condition, we are able to extend the main theorem to handle infinite direct sums in (1) and (2) (Lemma 2.2). The motivation for considering this case is a theorem of Kaplansky [9, Theorem 13.3] that for each submodule U of a free module M over a principal ideal ring with minimum condition there are sets of elements m_i of M and x_i of R such that $M = \sum \oplus Rm_i$ and $U = \sum \oplus Rx_i m_i$. It is not hard to see that each Rm_i can be chosen indecomposable. We show that every free (or, more generally, projective) left and right R-module has this *indecomposable simultaneous basis property* if and only if R is one of Nakayama's generalized uniserial rings (Theorem 2.5); and that if R is a principal ideal ring with minimum condition, then *every* (finite or infinite) direct sum decomposition $U = \sum_{i \in I} \oplus U_i$ of every submodule or homomorphic image U of a projective R-module M can be lifted to M (Theorem 2.4).

The proof of all the above results depends heavily on the hypothesis that M be projective. Concerning the necessity of this hypothesis we have the following partial result (Theorem 2.6): let M be a direct sum of cyclic torsion modules over a Dedekind domain R. The following are equivalent: (1) Every direct sum decomposition of every homomorphic image of M can be lifted; (2) Every direct sum decomposition of every submodule of M can be lifted; (3) Every primary component of M is homogeneous (i.e. the direct sum of mutually isomorphic cyclic modules). Note that condition (3) implies that each primary component is a *free* module over a ring of the form R/P^e.

Finally, we turn (in §3) to the question of uniqueness in suitably refined pairs of decompositions. The first result is: let $f: M$ onto $U = \sum_{i \in I} \oplus U_i$ be a homomorphism of R-modules for which there exist two *minimal liftings*

$$M = M_0 \oplus \left(\sum_{i \in I} \oplus M_i \right) = N_0 \oplus \left(\sum_{i \in I} \oplus N_i \right)$$

(that is, for $i \neq 0$, $f(M_i) = U_i = f(N_i)$ with no submodule other than M_i and N_i themselves mapping onto U_i; and $f(M_0) = 0 = f(N_0)$). Suppose, for each $i \neq 0$ that every minimal epimorphism of M_i onto itself is 1-1 (a condition satisfied whenever M_i is projective or noetherian). Then $M_0 \cong N_0$, and for each $i \neq 0$,

$M_i \cong N_i$ over U_i. A dual result is given concerning lifting decompositions of submodules.

1. **Finite decompositions.** We will call a left R-module U *semiprimary* if U is finitely generated and if the ring of multiplications of R on $U/\text{rad } U$ satisfies the left minimum condition (hence is semisimple with minimum condition). It is easy to check that a finitely generated R-module U is semiprimary if and only if $J(U/\text{rad } U) = 0$ for some 2-sided ideal J such that R/J satisfies the left minimum condition.

EXAMPLES 1.1. (1) ${}_RR$ is semiprimary if and only if the ring $R/\text{rad} R$ is semisimple with minimum condition. Such rings are often called *semiprimary* rings. In particular, a commutative ring is semiprimary if and only if it has only a finite number of maximal ideals.

(2) Every finitely generated module U over a semiprimary ring R is semiprimary: Since $U/\text{rad } U$ is contained in a (complete) direct product of simple modules, it follows that $(\text{rad} R)(U/\text{rad } U) = 0$. But by hypothesis $R/\text{rad} R$ satisfies the minimum condition.

(3) Over a commutative ring R every module U of finite (composition) length is semiprimary: if U is cyclic, say $U = R/J$, then U is a module over the semiprimary *ring* R/J. For the general case $U = Ru_1 + \cdots + Ru_n$ where $Ru_i \cong R/J_i$ we can either use Lemma 1.3 below or observe directly that $R/(J_1 \cap \cdots \cap J_n)$ is a ring with minimum condition.

MAIN THEOREM 1.2[2]. *Let* $f: M = M_1 \oplus \cdots \oplus M_n \twoheadrightarrow U = U_1 + \cdots + U_n$ *be an epimorphism of modules over a ring* R *with each* M_i *projective and each* U_i *semiprimary. Suppose, for each* i, *that there is an epimorphism of* M_i *onto* U_i. *Then there is a decomposition* $M = M'_1 \oplus \cdots \oplus M'_n$ *for which* $f(M'_i) = U_i$ *and* $M'_i \cong M_i$.

The proof will follow three lemmas. The second and third will provide methods of "exchanging" direct summands by means of the following easily verified statement: If R-modules $M = A \oplus B$ and an R-homomorphism $\phi: A \to B$ are given, then $M = (1 + \phi)A \oplus B$.

LEMMA 1.3. *Every homomorphic image of a semiprimary module is semiprimary. If* $U_1, \cdots, U_n$ *are semiprimary submodules of some module then so is* $U_1 + \cdots + U_n$.

Proof. For every homomorphism $f: U \to V$ of R-modules we have $f(\text{rad } U) \subseteq \text{rad } V$ [4, Proposition 2, p. 64]. Hence if $f(U) = V$, there is also an epimorphism $\bar{f}: U/\text{rad } U \twoheadrightarrow V/\text{rad } V$. If U is semiprimary there is a 2-sided ideal J

(2) The author is indebted to E. Matlis, D. Zelinsky, and the referee for many suggestions which led to the present version of this theorem.

such that R/J satisfies the minimum condition and $J(U/\text{rad}\, U) = 0$. Applying $\bar{f}$ shows that then $J(V/\text{rad}\, V) = 0$ so that V is semiprimary.

Note that if A_1 and A_2 are 2-sided ideals of R such that each R/A_i satisfies the left minimum condition, then so does the ring $R/(A_1 \cap A_2)$, for the left ideals of all three rings coincide with their left R-submodules, and there is an obvious R-monomorphism of $R/(A_1 \cap A_2)$ into $R/A_1 \oplus R/A_2$.

It is sufficient to prove the second assertion for the case $n = 2$. Furthermore, since $U_1 + U_2$ is a homomorphic image of $U_1 \oplus U_2$, it is sufficient to prove that $U = U_1 \oplus U_2$ is semiprimary. The first sentence of this proof implies that rad( ) is an additive (covariant) functor. Hence $\text{rad}\,(U_1 \oplus U_2 = \text{rad}\, U_1 \oplus \text{rad}\, U_2$ [5, Proposition 1, p. 19]. Letting $A_i = \text{ann}\,(U_i/\text{rad}\, U_i)$ we see that

$$(A_1 \cap A_2)(U/\text{rad}\, U) \cong (A_1 \cap A_2)(U_1/\text{rad}\, U_1 \oplus U_2/\text{rad}\, U_2) = 0$$

so that, since each R/A_i satisfies the left minimum condition, so does $R/(A_1 \cap A_2)$. Thus U is semiprimary.

LEMMA 1.4. *Let $f : M = P \oplus N \twoheadrightarrow U$ be an R-module epimorphism, with P projective and U semiprimary. Suppose that there is an epimorphism of P onto U. Then there is a homomorphism $\phi : P \to N$ such that $f(1 + \phi)P = U$.*

Proof. Consider, first, the case that P, N, and U are semisimple (P need not be projective here) and construct the diagram below as follows: let $f(P) = V$ and write $P = P_1 \oplus P_2$ where $P_1 = \ker(f : P \to V)$ and hence f is 1-1 on P_2. Then write $N = N' \oplus N_3$ where $N_3 = \ker(f : N \to U)$ and f is 1-1 on N'. Finally, let W be a submodule of $f(N')$ such that $U = V \oplus W$ and write $N' = N_1 \oplus N_2$ where $f(N_2) = W$ and f is 1-1 on N_2 (since it is 1-1 on N').

$$\begin{array}{ccccc} M = & P_1 \oplus & P_2 \oplus N_1 \oplus & N_2 \oplus N_3 & \quad (f \text{ is 1-1 on } P_2 \text{ and } N_2). \\ f \big\downarrow & & \big\downarrow & \big\downarrow & \\ U = & & V & \oplus W & \end{array} \tag{1}$$

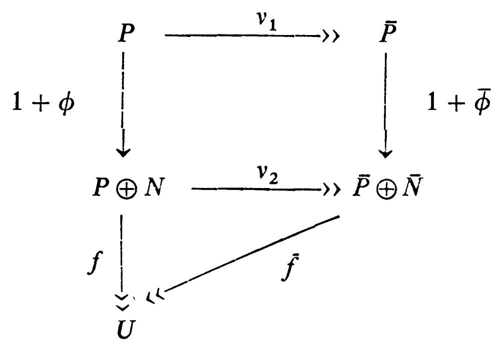
Since P is semisimple, the existence of an epimorphism $P \twoheadrightarrow U$ implies that U is isomorphic to a direct summand of P. Since further, U and hence V is *finitely generated*, the isomorphism $P_2 \cong V$ implies that W is isomorphic to a direct summand[3] of P_1. Hence there is an epimorphism ϕ of P_1 onto N_2 (which is $\cong W$); and if we define $\phi(P_2) = 0$ we obtain the desired $\phi : P \to N$ such that $f(1 + \phi)P = U$.

Now we consider the general case. If ϕ is any homomorphism: $P \to N$ and ν is the natural homomorphism: $U \to U/\text{rad}\, U$, then to show that $f(1 + \phi)P = U$ it is sufficient (since U is finitely generated) to show that $\nu f(1 + \phi)P = U/\text{rad}\, U$ [4, Corollary 3, p. 64]. Hence we suppose, from now on, that $\text{rad}\, U = 0$.

(3) This statement ,as well as the lemma and the theorem are actually false if V is not finitely generated. A counterexample is readily constructed with the help of (1).

In the diagram below, let $\bar{P} = P/(\text{ann } U)P$, $\bar{N} = N/(\text{ann } U)N$, let each ν_i denote the natural homomorphism, and let $\bar{f}$ be the map which makes the triangle at the bottom commute ($\bar{f}$ exists since $f(\ker \nu_2) = f(\text{ann } U)P \oplus f(\text{ann } U)N \subseteq (\text{ann } U)U = 0$). Note that $\bar{P}$, $\bar{N}$ and U are modules over the ring $R/\text{ann } U$ which is, by hypothesis, semisimple with minimum condition. Hence these modules are semisimple and there is a homomorphism $\bar{\phi}: \bar{P} \to \bar{N}$ such that $\bar{f}(1 + \bar{\phi})\bar{P} = U$. Since P is a projective R-module, there is a map $\phi: P \to N$ such that $\nu_2\phi = \bar{\phi}\nu_1$ (see the diagram again).

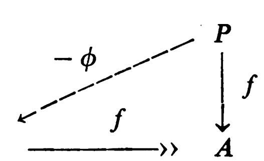
(2)



Commutativity of the diagram (2) now shows that $f(1 + \phi)P = \bar{f}(1 + \bar{\phi})\bar{P} = U$, completing the proof.

LEMMA 1.5. *Let $f: N \oplus P \twoheadrightarrow A$ be an R-module epimorphism with P projective and $f(N) = A$. Then there is a homomorphism $\phi: P \to N$ such that $f(1 + \phi)P = 0$.*

Proof. Since P is projective there is a $\phi: P \to N$ such that the following diagram commutes.



Then, on P, $f = -f\phi$. In other words, $f(1 + \phi)P = 0$.

Proof of the Theorem. Writing $f: M_1 \oplus (M_2 \oplus \cdots \oplus M_n) \twoheadrightarrow U_1 + (U_2 + \cdots + U_n)$ and recalling that a sum of semiprimary modules is again semiprimary (Lemma 1.3), we see that it suffices to prove the theorem for $n = 2$. In addition, recall that given any decomposition $M = A \oplus B$ and homomorphism $\phi: A \to B$ we also have $M = (1 + \phi)A \oplus B$. Note that $A \cong (1 + \phi)A$.

Let ν be the natural homomorphism of U onto $\bar{U}_1 = U/U_2$. By Lemma 1.3, $\bar{U}$ is semiprimary. Hence, by applying Lemma 1.4 to the situation $\nu f: M_1 \oplus M_2 \twoheadrightarrow \bar{U}_1$ we can exchange M_1 for a summand isomorphic to it (which we will again call M_1) such that $\nu f(M_1) = \bar{U}_1$. Then by Lemma 1.5 we can obtain a new M_2 such that $\nu f(M_2) = 0$, that is, $f(M_2) \subseteq U_2$.

We can now apply Lemma 1.4 to the situation

$$(1) \qquad f: f^{-1}(U_2) = M_2 \oplus (f^{-1}(U_2) \cap M_1) \twoheadrightarrow U_2$$

to obtain a new M_2 such that $f(M_2) = U_2$ (note that since $\phi: M_2 \to f^{-1}(U_2) \cap M_1$ is also a map: $M_2 \to M_1$ we still have $M = M_2 \oplus M_1$).

Let μ be the natural homomorphism of U onto $\bar{U}_2 = U/U_1$. Then applying Lemma 1.5 to μf: $M_1 \oplus M_2 \twoheadrightarrow \bar{U}_2$ we obtain $\mu f(M_1) = 0$ so that $f(M_1) \subseteq U_1$. Finally, by applying to $f^{-1}(U_1)$ the same argument used in (1) above we obtain $f(M_1) = U_1$ and complete the proof of the theorem.

The following theorem and its corollaries are intended to illustrate the type of information which can be obtained from the main theorem.

Recall that a projective module M over a commutative ring R has *rank* 1 if it is finitely generated, and if for every maximal ideal P of R, the R_P-module M_P is free of rank 1. We will call a module *local* if it has exactly one maximal submodule. Note that *if R is commutative, every cyclic module of finite (composition) length is a direct sum of local modules* [3, Corollary 1, p. 151].

THEOREM 1.6. *Let M be a direct sum of projective modules of rank 1 over a commutative ring R, and let H_i and K_i $(i = 1, \cdots, r)$ be submodules such that*

$$M/H_i \cong M/K_i = \textit{direct sum of a finite number of cyclic, local modules}$$

and such that the natural homomorphisms: $M \to \sum \oplus M/H_i$ and $M \to \sum \oplus M/K_i$ are epimorphisms. Then there is an automorphism σ of M such that $\sigma(H_i) = K_i$ for all i (in particular each $H_i \cong K_i$).

The proof will use two lemmas, the first of which establishes the theorem if M is itself projective of rank 1. An arithmetic proof of this lemma, for the case of Dedekind domains, is given in [7, Corollary 18.24].

LEMMA 1.7. *Let M be a projective module of rank 1 over a commutative ring R. Then*

(1) *Every submodule of M has the form EM for some unique ideal E of R.*

(2) *$M/EM \cong R/E$ whenever one of the sides is semiprimary.*

Proof. We first show that for every $f: M \to R$ and m, m' in M, we have

$$(3) \qquad (fm)m' = (fm')m.$$

To establish this it is sufficient to show, for every maximal ideal P of R, that $[(fm)m']_P = [(fm')m]_P$ [2, Corollary 1, p. 112]. Writing f_P for the map: $M_P \to R_P$ induced by f we can rewrite the desired equality in the form $(f_P m_P)m'_P = (f_P m'_P)m_P$. Since the R_P-module M_P is *free* of rank 1, we have reduced the proof of (3) to the case $M \cong R$, and hence to the case $M = R$. But here the result is obvious, since f is merely a multiplication by an element of R.

Let $M^* = \hom_R(M,R)$, and for a submodule N of M, let M^*N be the idea of R consisting of all sums of elements of the form $f(n)$ (f in M^*, n in N). Since M is projective of rank 1, $M^*M = R$ [2, Theorem 3, p.143]. Hence, if N is a submodule of M the previous paragraph shows that $N = (M^*M)N = (M^*N)M$. To obtain uniqueness of E ($= M^*N$) in (1), suppose $EM = FM$ and apply M^*, getting $E = F$.

To obtain (2) first note that for every ideal E of R,

(4) $$R/E \otimes_R M \cong M/EM \text{ (under } (r+E)\otimes m \to rm + EM),$$

and recall that a commutative ring is semiprimary if and only if it is semilocal (Example 1.1 (1)). If R/E is semiprimary, (4) shows that M/EM is projective of rank 1 over the semilocal ring R/E [2, Proposition 4, p. 142] and hence free of rank 1 over R/E [2, Proposition 5, p. 143]; that is, $M/EM \cong R/E$.

Finally we obtain (2) for the case that M/EM is semiprimary. It will be sufficient to show that M/EM is cyclic, say $\cong R/F$; for then by the previous paragraph $M/EM \cong R/F \cong M/FM$ showing $E = F$ (the annihilator of the left side is E). Since M is finitely generated, it is sufficient to show that M/EM *modulo its radical* is cyclic. The radical has the form GM/EM by (1). Hence it is sufficient to show that M/GM is cyclic. But the ring of multiplications of R on M/GM is isomorphic to R/G. Thus, R/G has minimum condition (hence is semiprimary) so that, by the previous paragraph, $M/GM \cong R/G$ is cyclic.

LEMMA 1.8. *Suppose there is an epimorphism*

$$f: M_1 \oplus \cdots \oplus M_n \twoheadrightarrow U = Rv_1 \oplus \cdots \oplus Rv_t \quad (Rv_i \cong R/E_i \neq 0)$$

where each M_i is projective of rank 1 and each E_i is contained in exactly one maximal ideal P_i. Then

(1) *Every finitely generated submodule of U is semiprimary.*

(2) *If $P_1 = P_2 = \cdots = P_t$, then $t \leqq n$.*

(3) *If $P_1, \cdots, P_t$ are all distinct, then $R(v_1 + \cdots + v_t) = Rv_1 \oplus \cdots \oplus Rv_t$.*

(4) *If $U = V_1 \oplus \cdots \oplus V_r$ where each V_j is a sum of Rv_i's, then there is a decomposition $U = Ru_1 \oplus \cdots \oplus Ru_n$ such that the projections u_{ij} of u_i in V_j satisfy $V_j = Ru_{1j} \oplus \cdots \oplus Ru_{nj}$.*

Proof. To establish (1) let $E = E_1 \cap \cdots \cap E_t$. Then the only maximal ideals which contain E are the P_i (any maximal ideal containing $E \supseteq E_1E_2 \cdots E_t$ would contain some E_i). Hence R/E is a semiprimary ring and every finitely generated submodule of the R/E-module U is semiprimary (Example 1.1).

To obtain (2) note that by (1) each $f(M_i)$ is semiprimary, and hence cyclic (Lemma 1.7). Hence U can be generated by n elements. This shows that the t dimensional vector space U/P_1U over the field R/P_1 is generated by n elements, so that $t \leqq n$.

To obtain (3) note that for $i \neq j$, $E_i + E_j = R$ (by hypothesis no maximal ideal contains both E_i and E_j), so that by one of the forms of the Chinese Remainder Theorem, $R/(E_1 \cap \cdots \cap E_t) \cong \sum_{i=1}^{t} \oplus R/E_i$.

To obtain (4), change the notation so that $P_1, \cdots, P_q$ are all of the *distinct* maximal ideals associated with the summands Rv_h, and for each k let

$$\left(\sum_{i=1}^{s(k)} \oplus Ru_{i1}^{k}\right) \oplus \left(\sum_{i=1}^{t(k)} \oplus Ru_{i2}^{k}\right) \oplus \cdots \qquad (u_{ij}^{k} \in V_j,\ u_{ij}^{k} = \text{some } v_h)$$

be all of these local summands associated with P_k. By (2) we have $s(k) + t(k) + \cdots \leqq n$. Hence we can build a $q \times n$ matrix of modules with the properties (i) each Ru_{ij}^{k} appears exactly once as an entry in row k, and (ii) the remaining entries in row k (if any) are zero. By (3), the sum of the modules in each *column* of this matrix will be cyclic, say Ru_i for column i, and will be generated by the sum u_i of the generators appearing in the column. The decomposition $U = Ru_1 \oplus \cdots \oplus Ru_n$ has the desired properties.

Proof of Theorem 1.6. First observe that given any epimorphism $f: M_1 \oplus M_2 \oplus \cdots \twoheadrightarrow U_1 \oplus U_2 \oplus \cdots$ with each $f(M_i) = U_i$, we have $\ker f$ $\sum_i \oplus \ker(f: M_i \twoheadrightarrow U_i)$.

Let U be a module isomorphic to $\sum \oplus M/H_i \cong \sum \oplus M/K_j$. By hypothesis there exist epimorphisms $f_i: M \twoheadrightarrow U$ such that $\ker f_1 = \bigcap_j H_j$ and $\ker f_2 = \bigcap_j K_j$, and $U = V_1 \oplus \cdots \oplus V_r$ where (p_j being the projection map) $p_j f_1(M) = V_j = p_j f_2(M)$ and $\ker p_j f_1 = H_j$, $\ker p_j f_2 = K_j$.

Consider, first, the case that M is the direct sum of a *finite* number n of projective modules of rank 1, say $M = M_1 \oplus \cdots \oplus M_n$. Letting p_j be the projection map: $U \to V_j$, we can apply Lemma 1.8 to obtain decompositions

$$\begin{array}{rcll} U & = & Ru_1 \oplus \cdots \oplus Ru_n & \qquad p_j(u_i) = u_{ij} \\ {\scriptstyle p_j} \big\downarrow & & & \\ V_j & = & Ru_{1j} \oplus \cdots \oplus Ru_{nj} & \qquad Ru_{ij} = R/E_{ij} \cong \text{semiprimary.} \end{array}$$

Now we apply the main theorem to obtain decompositions $M = M_1' \oplus \cdots \oplus M_n' = M_1'' \oplus \cdots \oplus M_n''$ with $f_1(M_i') = Ru_i = f_2(M_i'')$ and $M_i \cong M_i \cong M_i''$ (the required epimorphisms: $M_i \twoheadrightarrow Ru_i$ exist by Lemma 1.7). Let σ be any automorphism of M such that $\sigma(M_i') = M_i''$.

Since $p_j f_1(M_i') = Ru_{ij} = p_j f_2(M_i'')$ we see that

$$H_j = \ker(p_j f_1) = \sum_{i=1}^{n} \oplus \ker(p_j f_1 : M_i \twoheadrightarrow Ru_{ij}) = \sum_{i=1}^{n} \oplus E_{ij} M_{ij}'$$

(the extreme right-hand side by Lemma 1.7), and similarly $K_j = \sum_{i=1}^{n} \oplus E_{ij} M_{ij}''$. It now follows that $\sigma(H_j) = K_j$.

Consider, finally, the case that M is the direct sum of an infinite number of modules of rank 1, and suppose that U is the direct sum of n cyclic, local modules.

Then $M = L \oplus N$ where L is the direct sum of n modules of rank 1. By Lemma 1.7 there is an epimorphism: $L \twoheadrightarrow U$. Hence, by the main theorem, applied to $f_i: M = L \oplus N \twoheadrightarrow U \oplus 0$ $(i = 1,2)$ we obtain $M = L_i \oplus N_i$ with each $f_i(L_i) = U$, $f_i(N_i) = 0$, and $L_i \cong L$, $N_i \cong N$. We now have each $H_j = H_j' \oplus N_1$ and $K_j = K_j' \oplus N_2$ with $H_j' \subseteq L_1$ and $K_j' \subseteq L_2$. Identifying L_1 and L_2 we achieve a reduction to the finite case and complete the proof.

COROLLARY 1.9. *Let N_i be a submodule of a projective module M_i $(i = 1,2)$ over a Dedekind domain R, and suppose*

$$M_1/N_1 \cong M_2/N_2 = \text{finitely generated.} \tag{1}$$

Then $M_1 \cong M_2$ if and only if $N_1 \cong N_2$. When the conditions hold there is an isomorphism σ of M_1 onto M_2 such that $\sigma(N_1) = N_2$.

Proof. Since R is a Dedekind domain, every projective R-module is isomorphic to the (weak) direct sum of ideals of R [5, Theorem 5.3, p. 13 and p. 134]. Let J and K be direct sums of collections $\{J_i\colon i \in A\}$ and $\{K_i\colon i \in B\}$ respectively of nonzero ideals of R (we allow $J_i = J_h$ and $K_i = K_h$ for $i \neq h$). If the index set A is finite, then $J \cong K$ if and only if card A = card B and $\prod_{i \in A} J_i = u \prod_{i \in B} K_i$ ($\prod$ means "product in R") for some $u \neq 0$ in the field of fractions of R; while if A is infinite then J is *free* of rank card A, so that $J \cong K$ if and only if card A = card B [8]. One immediate consequence of these facts is the "cancellation" theorem: if J and K are direct sums of ideals of R and L is the direct sum of a *finite* number of ideals of R, then $J \oplus L \cong K \oplus L$ implies $J \cong K$.

If rank M_1 is infinite, then (1) shows that rank N_1 = rank M_1 so that $M_1 \cong N_1$ by the theorems in the previous paragraph. Hence either hypothesis $M_1 \cong M_2$ or $N_1 \cong N_2$ implies the other.

If rank M_1 is finite, first note that by Schanuel's Lemma(4) and (1) we have $M_1 \oplus N_2 \cong M_2 \oplus N_1$. The cancellation theorem quoted above now shows that either hypothesis $M_1 \cong M_2$ or $N_1 \cong N_2$ implies the other.

To complete the proof of the theorem, we now suppose $M_1 \cong M_2$ and wish to find an isomorphism σ of M_1 onto M_2 such that $\sigma(N_1) = N_2$. Let U be a module isomorphic to both M_1/N_1 and M_2/N_2, and for $i = 1,2$ choose an epimorphism $f_i: M_i \twoheadrightarrow U$ with $\ker f_i = N_i$. We can write U in the form $U = P \oplus T$ with T the torsion submodule of U and P projective [8].

Let $h_i: M_i \twoheadrightarrow P$ be the composition of f_i with the projection map: $U \twoheadrightarrow P$. Since P is projective, $M_i = P_i \oplus \ker h_i$ with $P_1 \cong P \cong P_2$. Our cancellation theorem

(4) SCHANUEL'S LEMMA. *Given two homomorphisms $f_i : M_i$ onto U $(i = 1,2)$ (modules over any ring) with each M_i projective, then $M_1 \oplus \ker f_2 \cong M_2 \oplus \ker f_1$.* For a proof, which seems to be available only in unpublished notes, let $T = \{(m_1, m_2) : f_1(m_1) = f_2(m_2)\}$. Then the homomorphism $(m_1, m_2) \to m_1$ sends T onto the projective module M_1 and its kernel is $(0, \ker f_2) \cong \ker f_2$. Hence $T \cong M_1 \oplus \ker f_2$. Similarly, $T \cong M_2 \oplus \ker f_1$.

then shows $\ker h_1 \cong \ker h_2$. Since $N_i = \ker f_i \subseteq \ker h_i$ we can define σ to be any isomorphism of P_1 onto P_2 and thereby reduce the proof to the case $P = 0$. U is now a torsion module, and hence the direct sum of a finite number of modules of the form R/P^e for various maximal ideals P and exponents e. Since these are cyclic, local modules, the theorem completes the proof of the corollary.

We can now easily obtain our completed version of the Steinitz-Chevalley theorem.

COROLLARY 1.10. *Let N be a submodule of a finitely generated projective module M over a Dedekind domain R, and suppose M/N is a torsion module. Then there exist decompositions*

$$(1) \qquad M = M_1 \oplus \cdots \oplus M_n \qquad (M_i \cong \text{ideal} \neq 0)$$

$$(2) \qquad N = E_1M_1 \oplus \cdots \oplus E_nM_n \qquad (E_i = \text{ideal} \neq 0)$$

where $M_1, \cdots, M_{n-1}$ can be chosen isomorphic to arbitrary nonzero ideals and the E_i can be prescribed provided only that $M/N \cong R/E_1 \oplus \cdots \oplus R/E_n$.

Before beginning the proof we note that neither of the restrictions at the end of the corollary can be omitted: given arbitrary nonzero ideals $K_1, \cdots, K_{n-1}$ there exists an ideal K_n, unique up to isomorphism, such that $M \cong \sum_{i=1}^n \oplus K_i$ (see the structure theorem quoted in the proof of 1.9); and (1) and (2) together with Lemma 1.7 show that $M/N \cong \sum \oplus R/E_i$. Observe, also, that by Lemma 1.8(4), M/N is a direct sum of n cyclic modules, i.e., ideals E_i exist such that $M/N \cong \sum_{i=1}^n \oplus R/E_i$.

Proof. Choose a decomposition $M = M'_1 \oplus \cdots \oplus M'_n$ with M'_i isomorphic to a prescribed ideal ($i = 1, \cdots, n-1$), and ideals E_i such that $M/N \cong \sum_{i=1}^n \oplus R/E_i$; and set $H = E_1M'_1 \oplus \cdots \oplus E_nM'_n$.

Since $M/H \cong M/N$ (Lemma 1.7) the previous corollary (or Theorem 1.6) shows that there is an automorphism σ of M such that $\sigma(H) = N$. But then $N = E_1\sigma(M'_1) \oplus \cdots \oplus E_n\sigma(M')$ so we have proved the corollary with $M_i = \sigma(M'_i)$.

REMARKS 1.11. A close look at the proof of the above corollary shows that it and Theorem 1.6 are actually equivalent when R is a Dedekind domain, M is finitely generated, and $r = 1$. Thus, essentially, the same proof as in the above corollary would establish the more general result (here $r = 2$): *let L and N be submodules of a projective module $M = M'_1 \oplus \cdots \oplus M'_n$ (R commutative) with each M'_i of rank 1. Suppose that $L + N = M$ and that both M/L and M/N are direct sums of cyclic, local modules. Then there exist decompositions*

$$M = M_1 \oplus \cdots \oplus M_n \qquad (M_i \cong M'_i),$$

$$L = E_1M_1 \oplus \cdots \oplus E_nM_n \qquad N = F_1M_1 \oplus \cdots \oplus F_nM_n.$$

However, here, and especially when more than two submodules $L, N, \cdots$ are involved, it is awkward to specify the exact amount of choice that exists for E_i and F_i. Therefore we have preferred to state the more abstract version given in Theorem 1.6.

For our final application of the main theorem, recall that we call a ring S *semi-primary* if $S/\operatorname{rad} S$ is semisimple with minimum condition. Two idempotents d, e of a ring R are called *isomorphic* ($d \cong e$) if there exist elements x, y in R such that $xy = d$ and $yx = e$. This is equivalent to saying $Rd \cong Re$ (as R-modules).

COROLLARY 1.12 (TO THE MAIN THEOREM). *Let f be a homomorphism of a ring R onto a semiprimary ring S, and suppose some orthogonal set $\bar{d}_1, \cdots, \bar{d}_n$ of idempotents of S can be lifted to an orthogonal set $d_1, \cdots, d_n$ of idempotents in R. Then every orthogonal set $\bar{e}_1, \cdots, \bar{e}_n$ of idempotents of S with $\bar{e}_i \cong \bar{d}_i$ can be lifted to orthogonal idempotents $e_1, \cdots, e_n$ in R with $e_i \cong d_i$.*

Proof. Let ν be the natural homomorphism of S onto $S/\operatorname{rad} S$ which is semi-simple with minimum condition, and note that for idempotents $\bar{d}, \bar{e}$ of S we have $\bar{d} \cong \bar{e}$ if and only if $\nu(\bar{d}) \cong \nu(\bar{e})$ (this follows either from the fact that $S\bar{d}$ is a projective cover for $S\nu(\bar{d})$ or from [Jacobson, *Structure of rings*, Proposition III.8.1]). Consequently we have the cancellation theorem: if $\bar{d}_1 + \bar{d}_2 \cong \bar{e}_1 + \bar{e}_2$ (all idpts. of S with $\bar{d}_2\bar{d}_1 = \bar{d}_1\bar{d}_2 = 0 = \bar{e}_1\bar{e}_2 = \bar{e}_2\bar{e}_1$) and $\bar{d}_1 \cong \bar{e}_1$, then $\bar{d}_2 \cong \bar{e}_2$.

Returning to the notation of the corollary, let $d_{n+1} = 1 - \sum_{i=1}^n d_i$, $\bar{d}_{n+1} = f(d_{n+1}) = 1 - \sum_{i=1}^n \bar{d}_i$, and $\bar{e}_{n+1} = 1 - \sum_{i=1}^n \bar{e}_i$. Then the preceding paragraph shows that $\bar{d}_{n+1} \cong \bar{e}_{n+1}$. Since S is a semiprimary R-module (scalar multiplication being given by $rs = f(r)s$), and since, for each i, there is an R-epimorphism: $Rd_i \twoheadrightarrow R\bar{e}_i$ ($\bar{d}_i \cong \bar{e}_i$) we can apply the main theorem to the situation $f: R = Rd_1 \oplus \cdots \oplus Rd_{n+1} \twoheadrightarrow S = R\bar{e}_1 \oplus \cdots \oplus R\bar{e}_{n+1}$ to obtain a new decomposition $R = Re_1 \oplus \cdots \oplus Re_{n+1}$ with $\{e_i\}$ an orthogonal set of idempotents, $e_i \cong d_i$, $f(Re_i) = R\bar{e}_i$ and $\sum_1^{n+1} e_i = 1$. Finally, from $\sum_1^{n+1} f(e_i) = f(1) = \sum_1^{n+1} \bar{e}_i$ and directness of $\sum \oplus R\bar{e}_i$ we conclude $f(e_i) = \bar{e}_i$ for each i.

2. **Simultaneous bases.** Let R be any ring. We will call a module homomorphism $f: M$ onto U a *minimal epimorphism* if no submodule of M other than M itself maps onto all of U. Given a homomorphism $f: M$ onto $U = \sum_{i \in I} \oplus U_i$ we will call a decomposition $M = M_0 \oplus \sum_{i \in I} \oplus M_i$ a *minimal lifting* if $f(M_0) = 0$ and if for each $i \neq 0$, $f(M_i) = U_i$ where each $f: M_i \to U_i$ is a minimal epimorphism.

LEMMA 2.1. *Let f: M onto U be a homomorphism of left modules over a ring R with the left minimum condition. Then f is a minimal epimorphism if and only if* $\ker f \subseteq (\operatorname{rad} R)M$.

Proof. Suppose that $\ker f \subseteq (\operatorname{rad} R)M$ and $f(L) = U$ for a submodule L of M. Then $M = L + (\operatorname{rad} R)M$. Substituting $L + (\operatorname{rad} R)M$ for M on the right-hand side we get $M = L + (\operatorname{rad} R)^2M = L + (\operatorname{rad} R)^3M = \cdots$. Since $\operatorname{rad} R$ is nilpotent this yields $M = L$ so that f is minimal.

Conversely, suppose that f is minimal, and consider the homomorphism $\bar{f}$ induced by the following diagram

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & U \\ \downarrow & & \downarrow \\ M/(\mathrm{rad}\,R)M & \xrightarrow{\ \bar{f}\ } & U/(\mathrm{rad}\,R)U \end{array}$$

where the vertical arrows represent natural homomorphisms and hence, by the preceding paragraph, are minimal epimorphisms. It follows that $\bar{f}$ is also a minimal epimorphism.

Since $M/(\mathrm{rad}\,R)M$ (which is a module over $R/(\mathrm{rad}\,R)$) is semisimple, $\ker \bar{f}$ is a direct summand of it, and minimality of $\bar{f}$ then shows $\ker \bar{f} = 0$. Hence $\ker f \subseteq (\mathrm{rad}\,R)M$.

A notion we will need in the next lemma is that of a *projective cover* of a module U, that is, a minimal epimorphism $f: P \to U$ where P is projective. It is easy to see that projective covers, when they exist, have the following uniqueness property: If $g: Q \to U$ is another projective cover of U, then there is an isomorphism $h: P$ onto Q such that $f = gh$.

LEMMA 2.2. *Let M be a projective left module over a ring R with the left minimum condition. Then every (finite or infinite) direct sum decomposition of every homomorphic image of M has a minimal lifting.*

Proof. Let $f: M$ onto $U = \sum_{i \in I} \oplus U_i$ be given. For each i let $g_i: P_i$ onto U_i be a projective cover of U_i (which exists since R satisfies the left minimum condition [1, Theorem P]), consider each P_i to be a submodule of the direct sum P of all of all of them, and let $g: P$ onto U be the direct sum of the maps g_i. By Lemma 2.1, $g: P \to U$ is a minimal epimorphism. Projectivity of M implies the existence of a homomorphism h, in the following diagram, such that $f = gh$.

$$\begin{array}{ccccc} M & \xrightarrow{\ h\ } & P & \qquad & P_i \\ & {\scriptstyle f}\searrow & \swarrow {\scriptstyle g\,(\text{minimal})} & & \downarrow {\scriptstyle g} \\ & & U & & U_i \end{array}$$

Then $g(h(M)) = U$ together with minimality of g implies that $h(M) = P$, and projectivity of P then implies $M = (\ker h) \oplus M'$ for some M'. Finally, the desired minimal lifting is $M = M_0 \oplus \sum_{i \in I} \oplus M_i$ where $M_0 = \ker h$ and M_i is the submodule of M' which is carried by h onto P_i (h is one-to-one on M').

REMARK. The two preceding lemmas are valid under the weaker hypothesis that R satisfy the minimum condition on *principal right* ideals (see 1, pp. 473–474 for the method of using T-nilpotence instead of nilpotence of the radical in Lemma 2.1 above) but we will not need the more general results here.

Let $U = \sum_{i \in I} \oplus U_i$ be a submodule of a module M over a ring R. By an *essential lifting* of the decomposition of U we mean a decomposition $M = M_0 \oplus \sum_{i \in I} \oplus M_i$ where each M_i is an essential extension of U_i $(i \neq 0)$. The following lemma is dual to the preceding one.

LEMMA 2.3. *Let M be an injective module over a left noetherian ring R. Then every direct sum decomposition of every submodule of M has an esential lifting.*

Proof. Let $U = \sum_{i \in I} \oplus U_i$ be a submodule of M and for each $i \in I$ choose an injective hull M_i of U_i in M [7, Theorem 54.13]. Note that the sum $\sum_{i \in I} M_i$ is direct: if there were a nontrivial relation $\sum m_i = 0$ ($m_i \in M_i$, only a finite number of terms nonzero) we could obtain another nontrivial relation $\sum rm_i = 0$ in which each $rm_i \in U_i$ by repeated use of the fact that M_i is an essential extension of U_i. Since all left ideals of R are finitely generated, $M' = \sum_{i \in I} \oplus M_i$ is injective [5, Chapter 1, Theorem 3.2] and hence $M = M_0 \oplus \sum_{\in I} \oplus M_i$ as desired.

THEOREM 2.4. *Let R be a left and right principal ideal ring with the left and right minimum condition, and let M be a projective R-module. Then every direct sum decomposition of every homomorphic image or submodule of M can be lifted, and the lifted decomposition can be chosen to be minimal and essential, respectively.*

Proof. The statement about homomorphic images is a special case of Lemma 2.2. To obtain the statement about submodules note that R is a quasi-frobenius ring [13, Theorem 16], and over such a ring every projective module is injective [10, Corollary 5.3]. Hence Lemma 2.3 completes the proof.

By *a generalized uniserial ring* we mean a ring R with the left and right R minimum condition such that every indecomposable left ideal Re as well as every indecomposable right ideal eR where $e^2 = e$ has exactly one composition series. The composition series of Re is then necessarily $Re \supset (\operatorname{rad} R)e \supset (\operatorname{rad} R)^2 e \supset \cdots \supset 0$; and every submodule of Re is cyclic, for if $x \in (\operatorname{rad} R)^t e$ but $x \notin (\operatorname{rad} R)^{t+1} e$, then $Rx = (\operatorname{rad} R)^t e$.

Recall that a module M has the *indecomposable simultaneous basis property* if for every submodule N of M there exist elements m_i in M and x_i in R such that $M = \sum \oplus Rm_i$, $N = \sum \oplus Rx_i m_i$ and each Rm_i is indecomposable.

THEOREM 2.5. *For a ring R the following are equivalent.*

(1) *Every projective*(5) *left R-module and every projective right R-module has the indecomposable simultaneous basis property.*

(2) *R is a generalized uniserial ring.*

Proof. (1) ⇒ (2). Condition (1) implies that every left and every right R-module is a direct sum of cyclic modules. Hence, by a theorem of Chase [6, Theorem 4.4], R satisfies the left and right minimum conditions. Condition (1) further implies

(5) This theorem remains true if *free* is substituted for *projective* at this point. The proof is the same in both cases.

that every left module is the direct sum of homomorphic images of indecomposable modules of the form Re ($e^2 = e \in R$), and a similar statement for right modules. But a theorem of Nakayama [12] asserts that among rings with left and right minimum condition only generalized uniserial rings have this property.

(2) ⇒ (1). Let N be a submodule of a projective left module M over the generalized uniserial ring R, and let f be the natural homomorphism of M onto $U = M/N$. Nakayama has shown [13, Appendix]; or [11] that U has a decomposition $U = \sum_{i \in I} \oplus Ru_i$ where each Ru_i is a homomorphic image of an indecomposable left ideal Re_i ($e_i^2 = e_i$). By Lemma 2.2 there is a minimal lifting of this decomposition of U, $M = M_0 \oplus \sum_{i \in I} \oplus M_i$. The kernel of the homomorphism: Re_i onto Ru_i is contained in $(\operatorname{rad} R)e_i$ and hence the map is a minimal epimorphism (Lemma 2.1). Uniqueness of the projective cover then shows $M_i \cong Re_i$, say, $M_i = Rm_i$ ($i \neq 0$).

Since $f(M_0) = 0$ and $f(Rm_i) = Ru_i$ for $i \neq 0$, it is easy to see that

$$N = \ker f = M_0 \oplus \sum_{i \in I} \oplus (Rm_i \cap \ker f)$$

(see the first paragraph of the proof of 1.13). Since every submodule of Re_i is cyclic, the same is true of Rm_i, say, $Rm_i \cap \ker f = Rx_i m_i$. A second application of Nakayama's theorem gives a decomposition $M_0 = \sum_{i \in J} \oplus Rm_i$.

Thus we have our indecomposable simultaneous basis:

$$M = \sum_{i \in I \cup J} \oplus Rm_i \quad \text{and} \quad N = \left(\sum_{i \in J} \oplus Rm_i \right) \oplus \left(\sum_{i \in I} \oplus Rx_i m_i \right).$$

THEOREM 2.6. *Let M be a direct sum of cyclic torsion modules over a Dedekind domain R. The following are equivalent.*

(1) *Every direct sum decomposition of every homomorphic image of M can be lifted.*

(2) *Every direct sum decomposition of every submodule of M can be lifted.*

(3) *Each primary component of M is homogeneous (i.e., the direct sum of mutually isomorphic cyclic modules).*

When the conditions hold the liftings in (1) *and* (2) *can be chosen to be minimal and essential respectively.*

Proof. Since the decomposition of M into the direct sum of its primary components induces primary decompositions of all the homomorphic images and submodules of M, it is sufficient to consider the case that M itself is primary, say with respect to the maximal ideal P.

Now suppose (3) holds. Then M is the direct sum of copies of R/P^e for some fixed e. In other words M is a free R/P^e-module. Since R/P^e is a principal ideal ring with minimal condition (Its only ideals are P^t/P^e where $0 \leqq t \leqq e$), conditions (1) and (2) and the supplementary statement are consequences of Theorem 2.4.

(1) ⇒ (3). Suppose that (3) is false, that is (since M is primary), suppose that M

has two nonisomorphic cyclic summands, say $M = \Sigma_{\in I} \oplus Rm_i$ where $Rm_1 \cong R/P^d$ and $Rm_2 \cong R/P^e$ $(1 \leqq d < e)$. (The index set I need not be countable.) Let $U = \Sigma_{i \in I} \oplus Ru_i$ where $Ru_i \cong Rm_i$ for $i \neq 2$ and $Ru_2 \cong R/P^d$.

Define $f: M$ onto U by

$$\left.\begin{array}{lll} f(m_i) = u_i & & (i \neq 1) \\ f(m_1) = u_1 + u_2 & & \end{array}\right\} \text{ so } \ker f \subseteq Pm_2.$$

To see that the given decomposition of U cannot be lifted, suppose $f(x_i) = u_i$ $(i = 1,2;\ x_i \in M)$. Then

$$f(x_1) = u_1 \Rightarrow x_1 = m_1 - m_2 + am_2 \qquad (a \in P),$$

$$f(x_2) = u_2 \Rightarrow x_2 = m_2 + bm_2 \qquad (b \in P),$$

so that, since $P^{e-1}m_1 = 0$, $P^{e-1}x_2 = P^{e-1}m_2 \ (\neq 0) = P^{e-1}x_1$. Thus $(Rx_2) \cap (Rx_1) \neq 0$ and the decomposition cannot be lifted.

(2) $\Rightarrow$ (3). Again suppose (3) is false and define M as above. Choose p in P but not in P^2. Then $U = Rm_1 \oplus R(m_1 + p^{e-d}m_2) \oplus \Sigma_{i \neq 1,2} \oplus Rm_i$ is an unliftable decomposition of a submodule of M.

3. **Uniqueness.** For definitions of *minimal lifting* and *minimal epimorphism* see the beginning of §2.

THEOREM 3.1. *Let $f: M$ onto $U = \Sigma_{i \in I} \oplus U_i$ be a homomorphism of R-modules for which two minimal liftings $M = M_0 \oplus \Sigma_{i \in I} \oplus M_i = N_0 \oplus \Sigma_{i \in I} \oplus N_i$ exist. Suppose, for each $i \neq 0$, that every minimal epimorphism of M_i onto itself is* 1-1 (*a condition satisfied whenever M_i is projective or satisfies the ACC*). *Then*

(1) *$M_0 \cong N_0$ and for $i \neq 0$ there exist isomorphisms q_i: M_i onto N_i such that the following diagrams commute*

$$\begin{array}{ccc} M_i & \xrightarrow{\ q_i\ } & N_i \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f} & \\ & U_i & \end{array} .$$

(2) *For every subset J of the index set $I \cup \{0\}$,*

$$\Big(M = \sum_{j \in J} \oplus M_j\Big) \oplus \Big(\sum_{j \in I \cup \{0\} - J} \oplus N_j\Big).$$

Proof. First we verify the assertion in parenthesis. If g maps M_i onto the *projective* module M_i, then $M_i = (\ker g) \oplus M'$ for some M'. Minimality of g then implies that $\ker g = 0$. On the other hand, if M_i satisfies the ascending chain condition, but g were not 1-1, then the infinite ascending chain

$$0 \subset \ker g \subset \ker g^2 \subset \ker g^3 \subset \cdots$$

would yield a contradiction.

Now let p_i be the projection map: $M \to M_i$ and q_i the projection map: $N \to N_i$, and fix an index $i \neq 0$. We now establish the diagram in (1).

Take $m_i \in M_i$ and write it in terms of the N's: $m_i = \sum_j q_j(m_i)$ (a finite sum). Applying f and using directness of $\sum_{j \in I} \oplus U_i$ we conclude

$$f(m_i) = fq_i(m_i) \qquad (i \neq 0) \tag{3}$$

and hence $U_i = f(M_i) = f(q_i(M_i))$. Thus $q_i(M_i)$ is a submodule of N_i which is mapped by f onto all of U_i. Minimality of $f: N_i \to U_i$ then implies that $q_i(M_i) = N_i$. Because of this and (3), the proof of the diagram in (1) will be complete if we can show that q_i is 1-1 on M_i.

To do this, observe that by symmetry $p_i q_i(M_i) = p_i(N_i) = M_i$ so that if we can show $p_i q_i$ to be a minimal epimorphism, then it will be 1-1 by hypothesis, and hence q_i will be 1-1. Hence take a submodule $L \subseteq M_i$ and suppose that $p_i q_i(L) = M_i$. Then $U_i = f(M_i) = fp_i q_i(L) = fq_i(L) = f(L)$ (by (3) and symmetry). But then minimality of $f: M_i \to U_i$ implies $L = M_i$ as desired.

Before showing $M_0 \cong N_0$ we establish (2). Let J be a subset of $I \cup \{0\}$ and let $K = (I \cup \{0\}) - J$. Suppose, for definiteness, that $0 \in K$. This is possible without loss of generality since we have shown that $M_i \cong N_i$ for $i \neq 0$ and hence the roles of M_i and N_i can be interchanged. Define

$$q^J: \sum_{j \in J} \oplus M_j = M^J \text{ onto } \sum_{j \in J} \oplus N_j = N^J \text{ by } q^J\left(\sum m_j\right) = \sum q_j m_j.$$

Since $0 \notin J$, q^J is an isomorphism (by the part of (1) which has already been established). Now consider the homomorphism

$$(q^J)^{-1}\left(\sum_{j \in J} q_j\right)$$

which maps (all of)M onto M^J. This homomomorphism is clearly idempotent and its kernel is $N^K = \sum_{k \in K} \oplus N_k$. Thus $M = M^J \oplus N^K$, and (2) is proved.

Finally, by (2), $M_0 \oplus M^I = M = N_0 \oplus M^I$ so that $M_0 \cong M/M^I \cong N_0$, completing the proof of the theorem.

We now state a theorem dual to the preceding one. However we omit the proof which is also dual to the previous one. Given a submodule U of a module M we will call a decomposition $M = M_0 \oplus \sum_{i \in I} \oplus M_i$ an *essential lifting* of a decomposition $U = \sum_{i \in I} \oplus U_i$ if each M_i $(i \neq 0)$ is an essential extension of U_i.

THEOREM 3.2. *Let $U = U_0 \oplus \sum_{i \in I} \oplus U_i$ be a submodule of a module M for which two essential liftings $M = M_0 \oplus \sum_{i \in I} \oplus M_i = N_0 \oplus \sum_{i \in I} \oplus N_i$ exist. Suppose, for each $i \neq 0$, that every monomorphism of M_i onto an essential submodule of itself is onto all of M_i (a condition satisfied whenever M_i is injective or satisfies the DCC). Then*

(1) $M_0 \cong N_0$; *and for* $i \neq 0$, $M_i \cong N_i$ *over* U_i.
(2) *For each subset* J *of the index set* $I \cup \{0\}$,

$$M = \left(\sum_{i \in J} \oplus M_j \right) \oplus \left(\sum_{j \in I \cup \{0\} - J} \oplus N_j \right).$$

The two theorems of this section contain the uniqueness of the projective cover and injective hull. (Let the given module be U, take $M_0 = 0, I$ a one-element set, and M projective or injective.) The following couniqueness and exchange properties are also immediate consequence of Theorems 3.1 and 3.2 (merely choose I to be a one-element set).

COROLLARY 3.3. *Let a module homomorphism* $f: M$ *onto* U *be given and suppose there exist two decompositions* $M = M_0 \oplus M_1 = N_0 \oplus N_1$ *where* $f(M_0) = f(N_0) = 0$ *and* $f: M_1 \to U$ *and* $f: N_1 \to U$ *are projective covers. Then* $M_0 \cong N_0$. *In fact* $M = M_0 \oplus N_1 = N_0 \oplus M_1$.

COROLLARY 3.4. *Let* U *be a submodule of a module* M *and suppose that there exist two decompositions* $M = M_0 \oplus M_1 = N_0 \oplus N_1$ *where* M_1 *and* N_1 *are injective hulls of* U *(in particular, both contain* U*). Then* $M_0 \cong N_0$. *In fact* $M = M_0 \oplus N_1 = N_0 \oplus M_1$.

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